1800's: Part I

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With the work of Euler much of the modern way of doing mathematics, sans the set theoretic clothing, is in place. It is also moving much quicker. Here we will take some shorter notes on the work done in the timespan 1800 - 1850 roughly. This is by the likes of Gauss, Fourier, Abel, Galois, Bolzano, Cauchy, Liouville and Dirichlet.

The things we are looking to pay attention to still are the definition of the reals, as Euler left us in the lurch when it comes to the definition of transcendentals. We are getting towards modern times and so the concepts that are being introduced are modern ones. We are looking out for how groups and topological spaces began to come into existance as well as modern notions of function, analysis and algebra.

Remark. Dates next to the names refer to the release of the read work, not the life of the person.

1 Gauss (1824)

Reading "General Investigations of Curved Surfaces (1825)" [GMH07]. My first remark is that this is quite modern in a couple of senses. No background, no axioms, no discussion. This is a modern research paper that assumes you are familiar with other contemporary research papers. I am not so familiar with how *his contemporaries* were treating, in particular calculus, this is something I would need to look into. This paper has nothing new in terms of foundations or ontology of geometry, it is entirely couched in the language that was popular at the time (which for me, is Euler, but I should see if that is accurate). Apparently the key references are Eulers books on differential and integral calculus. In addition the textbook by [Lac16].

I guess post Euler it is very clear how to go between algebra-functions and geometry. Another interesting thing that came up is that apparently Gauss loved Newton. He apparently emulated his enigmatic and unmotivated approach. We will see I suppose.

Definitions of curvature. After setting up some basic coordinates Gauss gives a nice little definition of continuity,

> A curved surface is said to possess continuous curvature at one of its points A, if the directions of all the straight lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing through A.

Essentially saying that all the tangent lines lie in the same plane. Gauss then goes on to explain how, if given a point on a curved surface one obtains a point on the unit sphere. The construction is to simply take the a unit length normal vector to the surface and then translate it to the origin. Thus also in this way curves and figures on our surface define curves and figures on the unit sphere. This leads to Gausses definition of "total or integral curvature"

> Thus, to each part of a curved surface inclosed within definite limits we assign a total or integral curvature, which is represented by the area of the figure on the sphere corresponding to it. From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the measure of curvature. The latter refers to a point of the surface, and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself; and hence it denotes the ratio of the infinitely small areas which correspond to one another on the curved surface and on the sphere.

Thus we have the curvature of a area on a curve defined and the curvature at the point is the limit of the ratio of the area on the curve and the area on the sphere. He goes on to find some formulas for this thing.

Flavours of topology. Gauss is really developing more of what we would call in a modern sense geometry I think. Anyway the two are closely related and there are interesting comments in here.

> two important points are to be considered, one when quantity alone is considered, the other when, disregarding quantitative relations, position alone is considered.
>
> When a surface is regarded, not as the boundary of a solid, but as a flexible, though not extensible solid, one dimension of which is supposed to vanish, then the properties of the surface depend in part upon the form to which we can suppose it reduced, and in part are absolute and **remain invariable, whatever may be the form into which the surface is bent.** To these latter properties, the study of which opens to geometry a new and fertile field, belong the measure of curvature and the integral curvature, in the sense which we have given to these expressions. To these belong also the theory of shortest lines, and a great part of what we reserve to be treated later. **From this point of view, a plane surface and a surface developable on a plane**, e. g., cylindrical surfaces, conical surfaces, etc., **are to be regarded as essentially identical**

What Gauss means by "developable" is apparently something like origami or having a net, in the sense of like primary school. That is it can be constructed from a rigid and flat material by gluing, folding, rolling and cutting, but not stretching.

Remark. (Lacroix) After examining this text to see how it differs from Eulers treatment there are a few things that are note worthy.

First he has basically the same conception of functions and calculus as Euler. He does however have a very strong reliance on the idea of taking limits, as opposed to Eulers less concrete ideas about just setting things to zero. The idea of limit is still naive. He defines

> The differential calculus is the finding the limit of the ratios of the simultaneous increment of the function, and of the variable on which it depends.

He also has a nice comment on the geometric vs the algebraic calculus

> Geometers first arrived at the differential calculus, which has since been exhibited under so many different points of view; but whatever the origins it will always depend on an analytic fact antecedent to any hypothesis, such as the behaviour of falling bodies; this fact is precisely that property which all functions possess, of admitting a limit in the ratio between their increments and that of the variable on which they depend.

So even though calculus originates in geometry it is more properly suited to analysis (algebra) because it really isnt about lines but increasing and decreasing ratios and this is a property that exists for all functions.

Remark. (Foundations) So geometry and number theory are somewhat unified by functions. At this time there are still different types of things. There is lines and numbers. One can go between them with functions and algebra, essentially by looking at the length of the lines. Note that this "taking the length" was not itself a function, as it is now, but merely a sort of philosophical change of perspective. The magic of set theory then is that it makes all of these things a single type and therefore allows us to deal with them all at once, and functions between geometry and algebra (like taking the length) in a single framework. I beleive that this really was the inovation of set theory, not being done before. Another thing to note is that theoretically we could have a multi-typed system that encompasses both instead of a uni-typed system, but we dont at this time.

2 Cauchy (1820 - 1845)

Cauchy (prefigured only by Lagrange) was a founder of group theory. He also made contributions to analysis with a new idea of limits, lets see.

2.1 Course of Analysis

Translated in [BS09] originaly published in 1821. Despite this translation stating

> Not only did Cauchy provide a workable definition of limits and a means to make them the basis of a rigorous theory of calculus, but also he revitalized the idea that all mathematics could be set on such rigorous foundations.

we will see that this is probably not true. Cauchy certainly aimed to bring a new taste of rigour to analysis, as he writes

> As for the methods, I have sought to give them all the rigor which one demands from geometry, so that one need never rely on arguments drawn from the generality of algebra.

He criticises the algebraist loose use of series

> We must also observe that they tend to grant a limitless scope to algebraic formulas, whereas, in reality, most of these formulas are valid only under certain conditions or for certain values of the quantities involved.

It is, as far as we can see, in the preliminaries chapter that Cauchy "defines" limits as follows

> When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the limit of all the other values.

As we have seen above (to my eyes) similar loose definitions of limit were already in use, indeed they were one of the major options in the raging debate of the 1700's. Cauchys "definition" seems no more rigerous than that in for instance Lacrouix. People agree with me. Now Cauchy did things with these limits that were new. His definition was not precise, but he developed an algebra of these limits that was I guess slightly more precise than previously. For example

> If k denotes a finite quantity different from zero and ϵ denotes a variable number that decreases indefinitely with the numerical value of α, the general form of infinitely small quantities of the first order is
> $$k\alpha(1 \pm \epsilon)$$

Notice that this is often cited as the entrance of the ϵ into the $\epsilon - \delta$ definition of limit. This definition of infinitely small quantities allows him to do some algebra. Cauchy also gives a very modern, however entirely prose definition of continuity

> Let $f(x)$ be a function of the variable x, and suppose that for each value of x between two given limits, the function always takes a unique finite value. If, beginning with a value of x contained between these limits, we add to the variable x an infinitely small increment α, the function itself is incremented by the difference
> $$f(x + \alpha) - f(x)$$
> , which depends both on the new variable α and on the value of x. Given this, the function $f(x)$ is a continuous function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference
> $$f(x + \alpha) - f(x)$$
> decreases indefinitely with the numerical value of α. In other words, the function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.

There is some controversy over this passage as some have suggested it was taken from Bolzano. Indeed the definition in Bolzano is almost verbatim the same.

Remark. This book does include the first uses of modulus and conjugate for complex numbers (that is these words).

Remark. So if Cauchy and Bolzano didnt have the precise definition of $\epsilon - \delta$ limits who did? Well some say Weirstrass, but from some stack exchange post I read apparently he never wrote it down, and all we have are the notes from his students in lectures, and to my eye they didnt look much different from what Cauchy has given us. This remains tbd.

2.2 Group Theory

I couldnt find a full translation of the relevant work, which is referenced by Galois (1815) or appeared later as a compendium (1845), however there is a partial translation and exposition in [Bar] and translated exerpts, that we will work from, in [Ste08, 13]. According to them

> One of several important features of Cauchy's paper was that he regarded such operations as mathematical objects in their own right, each labelled by a single capital letter, and capable of composition amongst themselves to form 'products' or 'powers'. This was the first time such notions, derived from arithmetic, had been extended to anything other than numerical quantities.

Indeed it seems that this does mark a new epoch in mathematics, adding to lines, numbers and functions the new possibility of *functions of functions*. First he has functions apply to "several quantities" which in his language [BS09] is a real number in our sense. But soon the function is to be really considered not as a function of the real numbers per se but of the symbols themselves

> In what follows, I will be concerned only with properties that the functions display by reason of their form, and not by reason of the set of values that the variables may take. Consequently, when it is a question of values equal amongst themselves that the function may take when one displaces the variables x, y, z, ..., one must always recall that these values are those that remain equal, whatever the values given to the variables x, y, z, ...

Note that this is a mix of his early and late manuscripts, and one can really see that the later writing is much more clear. Cauchys notion of a group is a "system of combined substitutions" he defines it here

> I will call derived substitutions all those that one may deduce from the given substitutions, multiplied one or several times by one another or by themselves in any order, and the given substitutions together with the derived substitutions will form what I will call a system of combined substitutions.

that is the substitutions generated on some given set of permutations. The rest of Cauchys work (that we could get our hands on) reads like a modern treatment of the symmetric group. He defines the order of a group element and proves Lagranges theorem.

Remark. (Contrast with Galois) I would say that Galois employed nothing new in the *ontological* side of group theory. All is present in Cauchy. What Galois did do is provide an application of this group from Cauchy to the roots of a polynomial. I mean in his first memoir, however I dont know about his other work.

Remark. [Bar] makes the following remark

> Cauchy's argument that the powers of (some group elemets) behave periodically, in a fashion reminiscent of how sets of roots of unity behave.

which is interesting as it makes it clear where other groups could be recognised.

3 Liouville (1844)

The original paper "On transcendental numbers" is provided here. I could not find a translation, so I used the "Claude" LLM. Since the paper is so short (only two pages) I will recount here the full translation gained (note that the document has a second unrelated part not provided here about something to do with Newton).

> M. LIOUVILLE verbally communicates to the Academy some remarks relating, 1st to very extensive classes of quantities whose value is neither rational nor even reducible to irrational algebraic radicals; 2nd to a passage from Newton's Principia where Newton calculates the action exerted by a sphere on an external point.
>
> **1.** To give examples of continued fractions from which one can rigorously demonstrate that their value is not the root of any algebraic equation
>
> $$f(x) = ax^n + bx^{(n-1)} + ... + gx + h = 0,$$

where $a, b, ..., g, h$ are integers, it suffices to recall that p/q and μ/q being two successive reduced fractions of the continued fraction that expresses the development of an incommensurable root x of this equation [1], the quotient of the incomplete μ, which comes after the reduced fraction μ/q, and serves to form the following, will end up (this results from a Lagrange formula, see the Berlin Memoirs, year 1768) by being, for very large values of q, constantly less than [2]

$$\pm\frac{df(p,q)}{q' f(p,q)dp},$$

an essentially positive expression where one supposes

$$f(p,q) = q^n f(p/q) = ap^n + bp^{(n-1)}q + ... + hq^n.$$

Abstraction made of signs, one will have all the more reason,

$$\mu < df(p,q)/(q^n p),$$

since $f(p,q)$ is an integer, equal at least to unity if one admits (which is permitted) that the equation $f(x) = 0$ has been cleared of all commensurable factor; $f(p,q) = 0$ would give in effect $f(p/q) = 0$. Now representing by $f'(x)$ the derivative of $f(x)$; the inequality above becomes

$$\mu < q^{(n-2)} f'(p/q).$$

Now, $f'(p/q)$ is a finite quantity which tends toward the limit $f'(x)$, as p/q tends toward the limit x. By designating by A a certain fixed number superior to this limit, one will thus be certain to have

$$\mu < Aq^{(n-2)}.$$

Thus the incomplete quotients of a continued fraction representing the root x of an algebraic equation of degree n, with rational coefficients, can never surpass the product of a certain number constant by the power (n - 2) of the denominator of the preceding reduced fraction. [3]

It will suffice to give to the incomplete quotients μ a mode of formation that makes them grow beyond the indicated term, to obtain continued fractions whose value cannot satisfy any algebraic equation properly so-called; this will happen, for example, if, starting from a first quotient incomplete whatever, one forms each of the following μ with the help of the reduced fraction p/q that precedes it, according to the law $\mu = q^s$, or even according to the law $\mu = q^m$, m being the index of the rank of μ.

Moreover, the preceding method, which was offered first, is neither the only nor even the simplest that one can employ. Let us add that there are also analogous theorems for ordinary series. We will cite in particular the series

$$\frac{1}{a} + \frac{1}{a^{1\cdot2}} + \frac{1}{a^{1\cdot2\cdot3}} + \cdots + \frac{1}{a^{1\cdot2\cdots m}} + \cdots,$$

a being a whole number.

[1]. Recall that a continued fraction is somthing of the form

$$b_0 + \frac{a_0}{b_1 + \frac{a_1}{b_2 + \frac{a_2}{b_3 + \cdots}}}$$

and that if the descent terminates the number is rational. If you allow the infinite continuation you get all irrational numbers (this is a modern statement). In some sense in Louivilles paper one can almost consider this the definition of a real number. Or a number is something given by a power series of rationals, something like that. What he shows is that algebraic numbers will satisfy a certain bound between their successive truncations of their continued (potentially infinite) fractions.

[2]. I cant really understand what hes talking about, I mean d is not the derivative as he uses the Newtonian notation, and I dont see why he would take the fourth constant in his list $a, b, c, ...(d), ...$ Regardless the punchline is what I have bolded [3].

Remark. So there we have it. The real numbers came from infinite series and continued fractions, essentially the idea of an (infinite) (not necissarily) decimal expansion. Its not from geometry at all but analysis in the sense of Euler. Of course geometric functions and incommensurability informed this in the early stages but ultimately it played little role.

4 Abel (1824)

Reading [Pes03, Appendix A] from 1824, it appears to be nothing special. He shows that quintics can be expresed in certain forms and then does a bunch of case work to show that if the parts are this or that you always end up with a contradiction to having a nice solution. Our interest is in his contributions to group theory, here there is only one mention of anything vaguely resembling this. He employs (although does not define) the idea of symmetric function, one that is not changed when permuting the inputs. He does not seem to have been the originator, nor is this really inherently group theoretic (although certainly it can be phrased that way).

According to this blog somewhere he showed that if an equation has solutions that are expressable as rational functions and those rational functions commute then the polynomial can be solved by radicals. This, when transformed into a statement about Galois groups, essentially says that the Galois groups of certain extensions are commutative. Hence the attribution. I dont think that this bears further investigating.

5 Galois (1846)

The relevant reference, containing translations is [GN11]. The paper was written in (the final draft) 1832 and published for the first time by Louiville in 1846. A good summary of the differences bewteen modern Galois theory and what is presented is written on Stack exchange. It is his postumous works that contain all the material for which he is now famous, in particular it is his "first memouri" that contains the so called "Galois theory". During his lifetime these works were rightly rejected by the academy with his reviewer Poisson writing

> His reasoning is neither clear enough nor well enough developed for us to have been able to judge its correctness, and we are in no position to give an idea of it in this report.

Jacques Tits also expresses this in 1982

> Is it shameful to admit that even today, having to judge the memoir in the form that Galois had given it, I would be pretty close to joining myself with the opinion expressed by Poisson?

It is really thanks to the commentaries and developments of later authors that his work was to become systematic enough to be useful and clear. It is obvious from his memoir that he had many original and profound ideas, however they were not clear to him or his readers. They were unpolished.

Groups Galois makes heavy use of the word "group". Did Cauchy before him? [GN11] argues that

> At first Galois used the word groupe as an ordinary French noun meaning 'group', 'set', 'collection'. It acquired a technical meaning only through repeated use. When the academy

> referees read his Premier Mémoire they would have had to infer any special meaning of the word from the proof of Proposition I, from the first scholium (see p. 116) that follows it, and (if they were not already stymied) from the regular use of it later in the paper. At that time Galois had not explained its meaning.

Galois does have several other remarks that clarify some of the properties that his "group" should have famously writing on the day before his dual

> if in such a group one has substitutions S and T , one is sure to have the substitution ST .

Fields Galois made no use of the concept of fields. His groups are associated to equations. He adjoins variables to equations. I guess in modern language the correspondence is given by getting a field from an equation by taking the splitting field. Everything in Galois revolves around a given equation however.

The Content A good commentary on the text that makes it slightly more understandable is [Edw12]. First Galois defines or clarifies what he means by rational. His equations have rational coefficients, or as he says

> When we thus agree to regard certain quantities as known, we shall say that we adjoin them to the equation which it is required to solve.

so we may regard a certain equation as "rational" under the assumption that say $\sqrt{2}$ has been adjoined to the equation, and this would mean that it is an equation with rational linear combinations of this irrational, or in modern terms in $\mathbb{Q}[\sqrt{2}]$.

Lets start with an equation

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

with some roots, $b_1, ..., b_k$, where the coefficients are certainly complex numbers, but likely restricted to rational numbers "adjoined" a couple of irrationals. The first main point of the paper is showing that "for any given polynomial f with, there is an irreducible polynomial g with with the property that" if we "adjoint" one of its roots to f (or in modern language the ground field) then f and g split into linear factors (it is the splitting field). "That is, Lemmas 2 and 3 imply a construction of a normal extension of K which is a splitting field of $f(x)$." Lets quote the two lemmas in full to see that its not easy to read this

> Lemma II. Given an arbitrary equation which has no equal roots, of which the roots are $a, b, c, ...$ one can always form a function V of the roots, such that none of the values that are obtained by permuting the roots in this function in all possible ways will be equal.
>
> Lemma III. The function V being chosen as is indicated in the preceding article, it will enjoy the property that all the roots of the proposed equation will be rationally expressible as a function of V .

honestly I read these statements and get brain freeze. Galois proposition one can be seen as both a construction and a definition of the group associated to an equation

> Theorem. Let an equation be given of which the m roots are $a, b, c, ...$ There will always be a group of permutations of the letters $a, b, c, ...$ which will enjoy the following property:
>
> 1. That every function of the roots invariant under the substitutions of this group will be rationally known;
>
> 2. Conversely, that every function of the roots that is rationally determinable will be invariant under the permutations substitutions.
>
> Substitutions are the passage from one permutation to another.

[Edw12] refrases this a little bit to make it clearer

> Proposition 1 (Revised). Let a [polynomial] be given of which the m roots are $a, b, c, \ldots.$ There will always be a group of permutations of the letters a, b, c, . . . which will enjoy the following property: 1. Every function of the roots $F(a, b, c, \ldots)$ that has a rationally known value has the same rationally known value when a substitution of this group is applied, and
>
> 2. conversely, every function of the roots $F(a, b, c, \ldots)$ that satisfies $F(a, b, c, \ldots) = F(Sa, Sb, Sc, \ldots)$ for all substitutions S in this group will have a rationally known value.

He then analyses how adjoining roots to the equation changes the associated group. [Edw12] explains that his proposition two is what we would now call the "fundamental theorem of Galois theory". The exposition is impenetrable. This provides his necessary and sufficient condition for an equation to be solvable, that the group is solvable. Note that this definition of the equation group makes it obvious that all equation groups are (in modern language) subgroups of the symmetric group. This turns out to be a theorem under the modern definition of (finite) groups.

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